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AN INTEGRAL EQUATION IN SYSTEMS THEORY

P. A. RUYMGAART
Gallaslaan 16, Leiderdorp, The Netherlands

T. T. SOONG†
Department of Civil Engineering, State University of New York at Buffalo,
Buffalo, NY 14260, U.S.A.

Abstract—This paper considers an integral equation of the Fredholm type which plays an important role in systems theory. The aim is to examine the uniqueness and differentiability of its solution. These properties are required in a number of system applications.

1. INTRODUCTION

The purpose of this note is to show that, under certain conditions, the integral equation

$$x(s,t) + \int_0^t k(s,u)x(u,t)\,\mathrm{d}u = y(s,t), \qquad s \in [0,t] \tag{1}$$

has a unique solution for $x(s,t)$ which is differentiable with respect to t. These properties are required, for example, in the derivation of the recursive equations of the Kalman–Bucy filter [e.g. 1] and in control theory [2].

2. PRELIMINARIES

Let $[0,T] \subset \mathbb{R}$ and $t \in [0,T]$. Let $L_2[0,t]$ be the Hilbert space of measurable functions f: $[0,t] \to \mathbb{C}$ with

$$\|f\|_t^2 = \int_0^t |f(s)|^2\,\mathrm{d}s < \infty,$$

where almost everywhere (a.e.) identical functions are identified. Then, with f and g in $L_2[0,t]$,

$$(f,g)_t = \int_0^t f(s)\overline{g(s)}\,\mathrm{d}s$$

is an inner product in $L_2[0,t]$. The overbar indicates "complex conjugate of".

We recall that $f_n \to f$ in $L_2[0,t]$ as $n \to \infty$, f_n and f in $L_2[0,t]$, means

$$\|f_n - f\|_t^2 = \int_0^t |f_n(s) - f(s)|^2\,\mathrm{d}s \to 0 \qquad \text{as } n \to \infty.$$

Let $C[0,t]$ be the Banach space of continuous functions f: $[0,t] \to \mathbb{C}$ with norm

$$\max_{s \in [0,t]} |f(s)|.$$

Here

$$f_n \to f \text{ in } C[0,t] \text{ as } n \to \infty, f_n \text{ and } f \text{ in } C[0,t]$$

means

$$f_n(s) \to f(s) \text{ as } n \to \infty, \text{ uniformly in } s \in [0,t].$$

†To whom all correspondence should be addressed.

We shall first consider equation (1) in $L_2[0, t]$. Let, at each $t \in [0, T]$,

$$\mathscr{A}_t: L_2[0, t] \to L_2[0, t]$$

be the Hilbert–Schmidt operator defined by

$$\forall f \in L_2[0, t], \quad \forall s \in [0, t]: (\mathscr{A}_t f)(s) = \int_0^t k(s, u) f(u)\, du,$$

where $k: [0, T]^2 \to \mathbb{R}$ is assumed to have the following properties:

(i) k is real-valued and continuous on its domain;
(ii) $\forall (s, t) \in [0, T]^2, k(s, t) = k(t, s)$;
and
(iii) $\forall t \in [0, T], \forall f \in L_2[0, t]$,

$$\int_0^t \int_0^t k(s, u) f(s) \overline{f(u)}\, ds\, du \geqslant 0.$$

Then $\mathscr{A}_t$ has the following properties:
(a) $\mathscr{A}_t$ is linear, bounded and even compact [e.g. 3];
(b) $\mathscr{A}_t$ is self-adjoint, since, as seen from properties (i) and (ii),

$$\begin{aligned}(\mathscr{A}_t f, g)_t &= \int_0^t \left[\int_0^t k(s, u) f(u)\, du \right] \overline{g(s)}\, ds \\ &= \int_0^t f(u) \left[\overline{\int_0^t k(u, s) g(s)\, ds} \right] du \\ &= (f, \mathscr{A}_t g)_t\end{aligned}$$

$\forall f$ and g in $L_2[0, t]$;
(c) $\mathscr{A}_t$ is semi-positive definite since, by property (iii),

$$\forall f \in L_2[0, t], \quad (\mathscr{A}_t f, f)_t = \int_0^t \int_0^t k(s, u) f(s) \overline{f(u)}\, ds\, du \geqslant 0.$$

In turn, these properties of $\mathscr{A}_t$ lead to the following properties for its spectrum [e.g. 3]:

(A) It follows from property (a) that the spectrum $\sigma(\mathscr{A}_t)$ of $\mathscr{A}_t$ consists of at most the number zero and a finite or denumerable set of eigenvalues, without accumulation points outside zero.
(B) Property (b) shows that $\sigma(\mathscr{A}_t) \subset \mathbb{R}$.
(C) As seen from property (c), $\forall t \in [0, T], \sigma(\mathscr{A}_t) \subset [0, \infty)$.

At each $t \in [0, T]$, let $\mathscr{E}_t: L_2[0, t] \to L_2[0, t]$ be the identity mapping

$$\forall t \in [0, T], \quad \forall f \in L_2[0, t], \quad \mathscr{E}_t f = f.$$

Denoting the functions $f: [0, t] \to \mathbb{R}$ by $f(\cdot, t)$ in the sequel, we consider, to each $y(\cdot, t) \in L_2[0, t]$, the following equation in $x(\cdot, t)$ with t fixed and $\lambda \in \mathbb{C}$:

$$\{\mathscr{A}_t - \lambda \mathscr{E}_t\} x(\cdot, t) = y(\cdot, t).$$

If $\lambda = -1$, we have equation (1), i.e.

$$\{\mathscr{A}_t + \mathscr{E}_t\} x(\cdot, t) = y(\cdot, t). \tag{2}$$

Since $-1 \notin \sigma(\mathscr{A}_t)$, it is well-known that [3]

$$\{\mathscr{A}_t + \mathscr{E}_t\}: L_2[0, t] \to L_2[0, t]$$

is a bicontinuous bijection, meaning that, at any $y(\cdot, t) \in L_2[0, t]$, there is exactly one $x(\cdot, t) \in L_2[0, t]$ satisfying equation (2). And, if to each $n \in \mathbb{N}$, $y_n(\cdot, t) \in L_2[0, t]$ and $x_n(\cdot, t)$ is the solution of equation (2) with $y_n(\cdot, t)$ on the r.h.s., we have, as $n \to \infty$,

$$y_n(\cdot\,,t)\to y(\cdot\,,t) \text{ in } L_2[0,t] \Leftrightarrow x_n(\cdot\,,t)\to x(\cdot\,,t) \text{ in } L_2[0,t].$$

Theorem 1 [e.g. 3]

The inverse $\{\mathscr{A}_t+\mathscr{E}_t\}^{-1}$: $L_2[0,t]\to L_2[0,t]$ of $\{\mathscr{A}_t+\mathscr{E}_t\}$ satisfies

$$\|\{\mathscr{A}_t+\mathscr{E}_t\}^{-1}\|_t \leqslant 1 \qquad \forall t\in[0,T]. \tag{3}$$

Proof. Since by assumption (iii), $0\leqslant(\mathscr{A}_t f,f)_t$ $\forall f\in L_2[0,t]$, we have

$$\begin{aligned}\|f\|_t^2 &\leqslant (\mathscr{A}_t f,f)_t+(f,f)_t\\ &=(\{\mathscr{A}_t+\mathscr{E}_t\}f,f)_t\leqslant \|\{\mathscr{A}_t+\mathscr{E}_t\}f\|_t\cdot\|f\|_t,\end{aligned}$$

the last inequality being the result of Cauchy's inequality. Hence,

$$\|f\|_t\leqslant\|\{\mathscr{A}_t+\mathscr{E}_t\}f\|_t. \tag{4}$$

Since $\{\mathscr{A}_t+\mathscr{E}_t\}$ is a bijection, to any $g\in L_2[0,t]$, there is an $f\in L_2[0,t]$, such that $f=\{\mathscr{A}_t+\mathscr{E}_t\}^{-1}g$ and conversely. Hence, equation (4) gives

$$\|\{\mathscr{A}_t+\mathscr{E}_t\}^{-1}g\|_t\leqslant\|g\|_t \qquad \forall g\in L_2[0,t],$$

showing equation (3), where the norm is understood to be an *operator-norm*.

Corollary

If $y(\cdot\,,t)\in L_2[0,t]$ and if $x(\cdot\,,t)$ is the solution of equation (1), then, since $x(\cdot\,,t)=\{\mathscr{A}_t+\mathscr{E}_t\}^{-1}y(\cdot\,,t)$,

$$\|x(\cdot\,,t)\|_t\leqslant\|y(\cdot\,,t)\|_t. \tag{5}$$

Let us now consider equation (1) in $C[0,t]$. Since for continuous f:$[0,t]\to\mathbb{C}$,

$$\int_0^t |f(s)|^2\,\mathrm{d}s<\infty,$$

we have, on ignoring the different topologies,

$$C[0,t]\subset L_2[0,t].$$

Let $\tilde{\mathscr{A}}_t$ be the restriction of $\mathscr{A}_t$ on $C[0,t]$. It is well-known [e.g. 3] that $\tilde{\mathscr{A}}_t$ maps $C[0,t]$ into $C[0,t]$ and that $\tilde{\mathscr{A}}_t$: $C[0,t]\to C[0,t]$ is also linear, bounded and even compact. Hence, the spectrum $\sigma(\tilde{\mathscr{A}}_t)$ of $\tilde{\mathscr{A}}_t$ consists of, at most, the number zero and a finite or denumerable set of eigenvalues, without accumulation points outside zero. Since clearly $\lambda\in\mathbb{C}$ is an eigenvalue of $\tilde{\mathscr{A}}_t$ if, and only if, it is an eigenvalue of $\mathscr{A}_t$, it follows that

$$\sigma(\tilde{\mathscr{A}}_t)=\sigma(\mathscr{A}_t).$$

Hence, $-1\notin\sigma(\tilde{\mathscr{A}}_t)$ and thus, if $\tilde{\mathscr{E}}_t$: $C[0,t]\to C[0,t]$ is the identity mapping in $C[0,t]$,

$$\{\tilde{A}_t+\tilde{\mathscr{E}}_t\}\colon C[0,t]\to C[0,t]$$

is a bicontinuous bijection [e.g. 3]. This gives the following result.

Theorem 2

At any $y(\cdot\,,t)\in C[0,t]$, there is exactly one $x(\cdot\,,t)\in C[0,t]$ satisfying equation (1). If to each $n\in\mathbb{N}$, $y_n(\cdot\,,t)\in C[0,t]$ and $x_n(\cdot\,,t)$ satisfying equation (1) with $y_n(\cdot\,,t)$ on the r.h.s., we have, as $n\to\infty$,

$$y_n(s,t)\to 0 \text{ uniformly in } s\in[0,t]$$

if, and only if, $x_n(s,t)$ does the same.

3. MAIN RESULT

We are now in a position to prove the main theorem.

Theorem 3

Let $D = \{(s,t) | 0 \leqslant s \leqslant t, t \in [0, T]\}$, let $k:[0,t]^2 \to \mathbb{R}$ have properties (i), (ii) and (iii) and let, at each $t \in [0, T]$, $y(\cdot, t) \in C[0,t]$ be continuously differentiable with respect to t. Then, if $x(\cdot, t)$ is the unique continuous solution of

$$x(s,t) + \int_0^t k(s,u)x(u,t)\,\mathrm{d}u = y(s,t), \qquad s \in [0,t], \tag{6}$$

we have

(α) $x(s,t)$ is bounded on D,
(β) $x(s,t)$ is continuous in (s,t) on D,
(γ) $\partial x(s,t)/\partial t$ exists on D

and

(δ) $\partial x(s,t)/\partial t$ is continuous in (s,t) on D.

Proof.

For part (α), it follows from equations (5) and (6) and Cauchy's inequality that, with $(s,t) \in D$,

$$\begin{aligned} |x(s,t)| &\leqslant |y(s,t)| + \left| \int_0^t k(s,u)x(u,t)\,\mathrm{d}u \right| \\ &\leqslant |y(s,t)| + \|k(s,\cdot)\|_t \cdot \|x(\cdot,t)\|_t \\ &\leqslant |y(s,t)| + \|k(s,\cdot)\|_t \cdot \|y(\cdot,t)\|_t. \end{aligned} \tag{7}$$

Since y and k are continuous on the compact set D, the r.h.s. of inequality (7) is bounded on D. Hence, there is a positive number M such that

$$|x(s,t)| \leqslant M \qquad \forall (s,t) \in D. \tag{8}$$

Consider now part (β). Let $(s,t) \in D$, $(s',t') \in D$ and suppose $s \leqslant s'$, $t \leqslant t'$. Then $s \in [0,t]$ and $s' \in [0,t']$ (see Fig. 1). And $x(s,t)$, $x(s,t')$ and $x(s',t')$ satisfy, respectively,

$$x(s,t) + \int_0^t k(s,u)x(u,t)\,\mathrm{d}u = y(s,t), \tag{9}$$

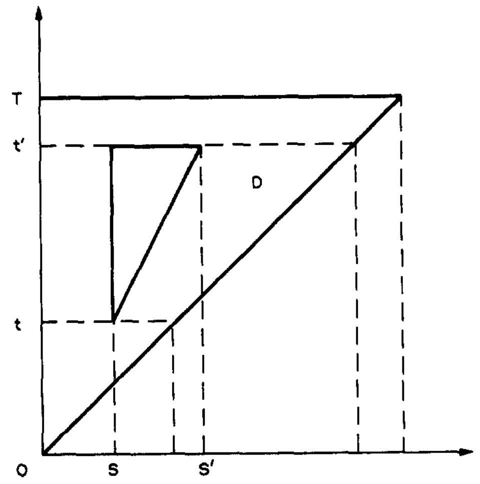

Fig. 1. The domain D.

$$x(s,t')+\int_0^{t'}k(s,u)x(u,t')\,du=y(s,t') \tag{10}$$

and

$$x(s',t')+\int_0^{t'}k(s',u)x(u,t')\,du=y(s',t'), \tag{11}$$

whereas

$$|x(s',t')-x(s,t)|\leqslant|x(s',t')-x(s,t')|+|x(s,t')-x(s,t)|. \tag{12}$$

We shall show that there is a neighborhood of (s, t) in D where the r.h.s. of inequality (12) is less than ϵ, ϵ being any positive number. First, as seen from equations (10) and (11),

$$\begin{aligned}|x(s',t')-x(s,t')|&\leqslant\int_0^{t'}|k(s',u)-k(s,u)|\cdot|x(u,t')|\,du+|y(s',t')-y(s,t')|\\&\leqslant M\int_0^{t'}|k(s',u)-k(s,u)|\,du+|y(s',t')-y(s,t')|.\end{aligned}$$

By virtue of the uniform continuity of k and y on their respective domains, there is a number $\delta_1>0$ such that both terms on the r.h.s. are less than $\epsilon/4$ if $|s-s'|<\delta_1$, uniformly in $t'\in[0,T]$.

Concerning the second term on the r.h.s. of inequality (12), it follows from equations (9) and (10) that

$$\begin{aligned}\{x(s,t')-x(s,t)\}&+\int_0^{t}k(s,u)\{x(u,t')-x(u,t)\}\,du\\&=\{y(s,t')-y(s,t)\}-\int_t^{t'}k(s,u)x(u,t')\,du. \end{aligned}\tag{13}$$

Apparently, $\{x(s,t')-x(s,t)\}$ satisfies an equation of the type given by equation (1). Since x is bounded [see part (α)] and since k and y are uniformly continuous on their domains, the r.h.s. of equation (13) tends to zero as $t'\to t$, uniformly in $s\in[0,T]$. Theorem 2 thus shows that there is a number $\delta_2>0$ such that $|t-t'|<\delta_2$ implies

$$|x(s,t')-x(s,t)|<\epsilon/2 \text{ uniformly in } s\in[0,t].$$

This completes the proof of part (β).

For part (γ), let us replace $\partial x(s,t)/\partial t$ by $\tilde{x}(s,t)$. Formal differentiation of equation (1) with respect to t yields

$$\tilde{x}(s,t)+\int_0^{t}k(s,u)\tilde{x}(u,t)\,du=\frac{\partial}{\partial t}y(s,t)-k(s,t)x(t,t),\qquad s\in[0,t], \tag{14}$$

where $x(t,t)$ is the solution of equation (1) at $s=t$. We may look upon equation (14) as an integral equation in $\tilde{x}(s,t)$. It is again of the type given by equation (1). Because of the continuity of its r.h.s., equation (14) has a unique solution $\tilde{x}(\cdot,t)$ in $C[0,t]$ at all $t\in[0,T]$.

Let $(s,t)\in D$ and suppose $0\leqslant t\leqslant t'\leqslant T$. Equations (9), (10) and (14) and mean-value theorems give

$$\begin{aligned}&\left\{\frac{x(s,t')-x(s,t)}{t'-t}-\tilde{x}(s,t)\right\}+\int_0^{t}k(s,u)\left\{\frac{x(u,t')-x(u,t)}{t'-t}-\tilde{x}(u,t)\right\}du\\&=\left\{\frac{y(s,t')-y(s,t)}{t'-t}-\frac{\partial}{\partial t}y(s,t)\right\}-\left\{\frac{1}{t'-t}\int_t^{t'}k(s,u)x(u,t')\,du-k(s,t)x(t,t)\right\}\\&=\left\{\frac{\partial}{\partial t''}y(s,t'')-\frac{\partial}{\partial t}y(s,t)\right\}-\{k(s,u'')x(u'',t')-k(s,t)x(t,t)\},\qquad t<t''<t',\ t<u''<t'.\end{aligned}\tag{15}$$

Equation (15) again may be seen as an integral equation of the type (1), this time in

$$\frac{x(s, t') - x(s, t)}{t' - t} - \tilde{x}(s, t).$$

Since $\partial y/\partial t$, k and x are uniformly continuous on their respective domains, the r.h.s. of equation (15) tends to zero as $t' \to t$, uniformly in $s \in [0, t]$. Hence, on account of Theorem 2,

$$\frac{x(s, t') - x(s, t)}{t' - t} - \tilde{x}(s, t) \to 0 \qquad \text{as } t' - t \to 0.$$

In other words, we have shown $\tilde{x}(s, t) = \partial x(s, t)/\partial t$ and thus part (γ). Moreover, it is seen that $\partial x(s, t)/\partial t$ satisfies equation (14).

Finally, for part (δ), the r.h.s. of equation (14) is continuous on D since $\partial y(s, t)/\partial t$ and $k(s, t)x(t, t)$ are continuous on D by virtue of part (β). Hence, again by virtue of part (β), $\tilde{x}(s, t) = \partial x(s, t)/\partial t$ is continuous on D as a solution of equation (14).

The proof of Theorem 3 is thus complete.

In closing, let us remark that the main result is given for a scalar equation. It is not difficult to generalize it to a system of equations [1].

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